Coefficient estimates for whole-plane SLE processes

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Abstract

Loewner introduced his famous differential equation in 1923 in order to solve Bieberbach conjecture for n=3. His method has been revived in 1999 by Ode Schramm who introduced Stochastic Loewner processes which happened to open many doors in statistical mechanics. The aim of this paper is to revisit Bieberbach conjecture in the framework of SLE and more generally Lévy processes. This has lead to astonishing results and conjectures.

1 Introduction

Let $f(z) = \sum_{n\geq 0} a_n z^n$ be a holomorphic function in the unit disc $\mathbb{D}$. We further assume that the function f is injective: what can be then said about the coefficients a_n? An trivial observation is that $a_1 \neq 0$ and Bieberbach [1] proved in 1916 that

$$|a_2| \leq 2|a_1|.$$

In the same paper he conjectured that

$$\forall n \geq 2,\ |a_n| \leq n|a_1|,$$

guided by the intuition that the function (called from that time Koebe function)

$$f(z) = \sum_{n\geq 2} n z^n,$$

which is a holomorphic bijection between $\mathbb{D}$ and $\mathbb{C}\backslash(-\infty, -1/4]$, should be extremal. This conjecture was proved in 1984 by De Branges [2]: this proof was made possible by the addition of a new idea (an inequality of Askey and Gasper) to a lot of methods and strategies that have been developed along almost a century of efforts.

It is largely accepted that the first main contribution to the proof of Bieberbach conjecture is the proof [9] by Loewner in 1923 that $|a_3| \leq 3|a_1|$. De Branges' proof indeed uses Loewner's idea in a crucial way as well as many contributors to the proof did in the period. But Loewner's idea goes far beyond Bieberbach conjecture: Oded Schramm [12] revived Loewner's method in 1999, introducing randomness in it, and obtaining as a consequence a unified way to understand many questions in statistical mechanics. This theory is now called the theory of SLE (for Schramm-Loewner, initially Stochastic Loewner evolution) processes.

The aim of the present paper is to revisit Bieberbach conjecture in the framework of SLE processes, that is to study what can be said about the coefficients of univalent functions coming from these processes. The main tool traditionally used for dealing with SLE processes is Ito calculus; by contrast, our approach is completely elementary, the main tools being Markov property for brownian motion and the expression for its characteristic function. This elementary approach allowed us to generalize considerably the validity of the results, namely in the framework of Loewner evolutions driven by Lévy processes.

In the first part we will outline the proof by Bieberbach of the case $n = 2$ and Loewner's proof for $n = 3$. Actually Loewner's method covers the case $n = 2$ but we keep Bieberbach's proof because it allows us to introduce basic notions needed to understand facts around this conjecture. We finish this first part by a brief account of post-Loewner steps in the proof of Bieberbach conjecture.

In the second part we will introduce SLE processes (in the terminology of Schramm, whole-plane SLE processes) and their generalizations using Lévy processes and study the coefficients (which are random variables) of the associated univalent functions.

The main result of this paper is the following surprisingly "universal"

Theorem 1.1. *Let (f_t) be a Lévy-Loewner process with Lévy symbol η (for the definitions see below) and*

$$f_0(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Then for $n \leq 20$ if $\eta_1 = 0,\ 1,\ 3$ we have respectively

$$E(|a_n|^2) = n^2,\ n,\ 1.$$

Before we come to the details, let us comment this statement. The case $\eta_1 = 0$ is obvious since it coincides with the Koebe function. Notice that in this case, by Schoenberg correspondence, η_2 must be equal to 0. The cases $\eta_1 = 1, 3$ correspond in the SLE case to $\kappa = 2, 6$ which are known to be connected respectively to self-avoiding random walks and to critical percolation. Is the result connected with these facts?

Also our proof is heavily computer assisted: if one gives us a number n then, waiting long enough (and this becomes very long if one approaches 20), we can prove the result. We predict that the theorem is valid for all values of n but we have no proof yet.

2 A brief history of the Bieberbach conjecture

2.1 Proof for $n = 2$ (Bieberbach, 1916)

First of all, let us introduce the normalized class

$$S = \{f : \mathbb{D} \to \mathbb{C} \,\text{holomorphic and injective}\,;\ f(0) = 0,\ f'(0) = 1\}.$$

Bieberbach conjecture is clearly equivalent to $|a_n| \leq n$, $n \geq 2$ for $f \in S$. A related class of normalized functions is

$$\Sigma = \left\{ f : \Delta = \overline{\mathbb{C}} \backslash \overline{\mathbb{D}} \to \overline{\mathbb{C}} \,\text{holomorphic and injective}\,;\ f(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n} \text{ at } \infty \right\}.$$

The application $f \mapsto F$ where $F(z) = 1/f(1/z)$ is clearly a bijection from S onto Σ', the subclass of Σ consisting in functions that do not vanish in Δ. A simple application of Stokes formula shows that if $f \in \Sigma$ then, denoting by $|B|$ the Lebesgue measure (area) of the borelian subset B of the plane,

$$|\mathbb{C} \backslash f(\Delta)| = \pi \left(1 - \sum_{n \geq 1} n |b_n|^2 \right).$$

Since area is a positive number, a consequence of this equality is that $|b_1| \leq 1$. If we apply this inequality directly to the function F in Σ' coming from $f \in S$, one does not obtain anything conclusive. The idea of Bieberbach is to apply this inequality to an odd function in S.

Let $f \in S$ then $z \mapsto f(z)/z$ does not vanish in the disc and thus it has a unique holomorphic square root g which is equal to 1 at 0. Then, $f(z^2) = h(z)^2$, where $h(z) = zg(z^2)$ is still in the class S but is moreover odd. This establishes a bijection $(f \mapsto h)$ between S and the set of odd functions in S. Now if $f(z) = z + a_2 z^2 + a_3 z^3 + \ldots$ belongs to the class S then it is easy to see that $h(z) = z + \frac{a_2}{z^3} + O(z^5)$ and that the associated $H \in \Sigma$ satisfies

$$H(z) = 1/h(1/z) = z - \frac{a_2}{2z} + \ldots$$

By area theorem,

$$|\frac{a_2}{2}| \leq 1 \Rightarrow |a_2| \leq 2.$$

2.2 Proof for $n = 3$ (Loewner [9], 1923)

Replacing $f(z)$ by $f(rz)$ with $r < 1$ but close to 1, one sees that it suffices to prove the estimate for conformal mappings onto smooth Jordan domains containing 0. Consider such a domain Ω and let $\gamma : [0, t_0] \to \mathbb{C}$ be a parametrization of its boundary. We consider then $\Gamma : [0, \infty) \to \mathbb{C}$ a Jordan arc joining $\gamma(0) = \gamma(t_0)$ to ∞ inside the outer Jordan component. We then define

$$\Lambda(t) = \gamma(t), t \leq t_0, = \Gamma(t - t_0),\ t \geq t_0,$$

and define for $s > 0$,

$$\Omega_s = \mathbb{C} \backslash \Lambda([s, \infty)).$$

It is a simply connected domain containing 0 and we can thus consider its Riemann mapping $f_s : \mathbb{D} \to \Omega_s$, $f_s(0) = 0, f_s'(0) > 0$. By Caratheodory convergence theorem, f_s converges as $s \to 0$ to f, the Riemann mapping of Ω. We may assume without loss of generality that $f'(0) = 1$ and, by changing time if necessary, that $f_s'(0) = e^s$.

The key idea of Loewner is to use the fact that the sequence of domains Ω_s is increasing, which translates into the fact that

$$\Re \left(\frac{\frac{\partial f_t}{\partial t}}{z \frac{\partial f_t}{\partial z}} \right) > 0$$

or, equivalently, that $\Re \left(\frac{\frac{\partial f_t}{\partial t}}{z \frac{\partial f_t}{\partial z}} \right)$ is the Poisson integral of a positive measure, actually a probability measure because of the normalization $f_t'(0) = e^t$. Now the fact that the domains Ω_t are slit domains implies that for every t this probability measure must be the Dirac mass at $\lambda(t) = f_t^{-1}(\Lambda(t))$. Even if it is not needed for the proof, it is worthwhile to notice that λ is a continuous function. We say that the process Ω_s is driven by the function λ, in the sense that (f_s) satisfies the Loewner differential equation

$$\frac{\partial f_t}{\partial t} = z \frac{\partial f_t}{\partial z} \frac{\lambda(t) + z}{\lambda(t) - z}. \tag{1}$$

To finish Loewner's proof we extend both sides of the last equation as power series and simply identifiy the coefficients. This leads to, where we have put $f_t(z) = e^t(z + a_2 z^2 + a_3 z^3 + \ldots)$:

$$\dot{a_2} - a_2 = 2\overline{\lambda},$$

$$\dot{a_3} - 2a_3 = 4a_2\overline{\lambda} + 2\overline{\lambda}^2$$

(to simplify the notations, we indicate t-derivative with a dot). The first differential equation is easily solved, giving

$$a_2(t) = -2e^t \int_t^\infty \overline{\lambda}(s) e^{-s} ds$$

and a new proof of the case $n = 2$. Once a_2 is known one can solve the second equation, leading to

$$a_3(t) = -4e^{2t}\int_t^\infty e^{-2s}a_2(s)\overline{\lambda}(s)ds - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

We simplify this expression by noticing that the first integral is of the form $\frac{1}{2}\int_t^\infty uu'$ where $u(s) = e^{-s}a_2(s)$. The formula for a_3 then simplifies to

$$a_3(t) = 4e^{2t}\left(\int_t^\infty \overline{\lambda}(s)e^{-s}ds\right)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

Before we continue with the proof we notice that, by considering $e^{-i\alpha}f(e^{i\alpha z})$, it suffices to prove that $\Re(a_3) \leq 3$. To this aim we write $\lambda(s) = e^{i\theta(s)}$. Using furthermore that $\cos(2\theta) = 2\cos^2(\theta) - 1$ and the fact that, by Cauchy-Schwarz inequality,

$$\left(e^t\int_t^\infty e^{-s}\cos\theta(s)ds\right)^2 \leq e^t\int_t^\infty e^{-s}\cos^2\theta(s)ds\,,$$

we get

$$\begin{aligned}
\Re(a_3) = 4e^{2t}&\left(\int_t^\infty e^{-s}\cos\theta(s)ds\right)^2 \\
&- 4e^{2t}\left(\int_t^\infty e^{-s}\sin\theta(s)ds\right)^2 - 2e^{2t}\int_t^\infty e^{-2s}\cos\left(2\theta(s)\right)ds \\
\leq 4&\int_t^\infty \left(e^{t-s} - e^{2(t-s)}\right)\cos^2\theta(s)ds + 1 \\
\leq 4&\int_t^\infty \left(e^{t-s} - e^{2(t-s)}\right)ds + 1 = 3.
\end{aligned}$$

It is remarkable that the Loewner method can be reversed: given a continuous function λ (or more generally a regulated function) from $[0,\infty)$ to the unit disk then the Loewner equation (1) has a unique solution $f_t(z)$ which is the Riemann mapping of a domain Ω_t, and the corresponding family is increasing in t. Notice that it is not true in general that the obtained domains are slit-domains.

In 1999, Oded Schramm had the intuition to take

$$\lambda(t) = e^{i\sqrt{\kappa}B_t}$$

where B_t is a standard one dimensional brownian motion, and this leads to the very powerful theory of *SLE* (Schramm-Loewner evolution) processes. We will come to these processes in the next paragraph.

2.3 Bieberbach conjecture after Loewner

The next milestone after 1923 and Loewner theorem is 1925 and the proof by Littlewood [7] that in the class S, $|a_n| \leq en$. In 1931, Dieudonné [3] has proven the conjecture for functions with real coefficients. In 1932, Littlewood and Paley [8] have proven that the coefficients of an odd function in S are bounded by 14 and they conjectured that the best bound is 1, a conjecture that implies Bieberbach's. This conjecture was disproved by Fekete and Szego [4] in 1933 for $n = 5$. In 1935, Robertson [11] stated the weaker conjecture

$$\sum_{k=1}^{n} |a_{2k+1}|^2 \leq n,$$

which also implies the Bieberbach conjecture. The next milestone is due to Lebedev and Milin [6]: it had already been observed by Grunsky [5] in 1939 that the logarithmic coefficients γ_n defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n$$

can easily be estimated. In the sixties, Lebedev and Milin [6] have shown, through three inequalities, how to pass from these estimates to estimates for f. This allowed Milin [10] to prove $|a_n| \leq 1.243n$ and consequently he stated what has become known as Milin conjecture:

$$\sum_{m=1}^{n} \sum_{k=1}^{m} \left(k|\gamma_k|^2 - \frac{1}{k} \right) \leq 0.$$

It should be noticed that $\gamma_n = 1/n$ for the Koebe function but the stronger conjecture $|\gamma_n| \leq 1/n$ is false, even as an order of magnitude. It happens that Milin $\Rightarrow$ Robertson $\Rightarrow$ Bieberbach, and De Branges actually proved Milin conjecture.

3 Coefficient estimates for SLE_κ

Whole-plane SLE_κ is the Loewner process (as defined in the last section) driven by the function

$$\lambda(t) = e^{i\sqrt{\kappa}B_t}$$

where B_t is a standard one-dimensional brownian motion. For such a process we will call a_n the Taylor coefficients of $f = f_0$. These coefficients are now random variables: their laws seem to be out of reach so we will restrict our study to the computations of the expectations of a_n and $|a_n|^2$ (from which we may of course deduce the variance). The computation of the expectation will lead to the computation of $E(f_t(z))$, while the computation of the variances will lead to the computation of the expectations of the integral means

$$\int_0^{2\pi} \left| f_t^{(n)}(re^{i\theta}) \right|^2 d\theta.$$

As we shall see, our elementary approach (no Ito calculus) only uses two properties of Brownian motions:
- Stationarity: if $0 \leq s < t$ then $B_t - B_s$ obeys a law depending only on $t - s$;
- Markov property: if $0 \leq s < t$ then $B_t - B_s$ is independent of B_s.

An important class of processes satisfying these two properties is a subset of the class of Lévy processes: these processes $(L_t)_{t\geq 0}$ are such that their characteristic functions are of the form

$$E(e^{i\xi L_t}) = e^{-t\eta(\xi)}$$

where η, called the Lévy symbol, satisfies some Bochner type condition and

$$\eta(-\xi) = \overline{\eta(\xi)}.$$

An important sub-class of Lévy processes is the class of $\alpha \in (0, 2]$-stable processes: these are the processes whose Lévy symbol are

$$\eta(\xi) = \frac{\kappa}{2}|\xi|^\alpha.$$

We have chosen the normalization $\kappa/2$ so that this process is SLE_κ when $\alpha = 2$: in the case of a general Lévy process the constant κ corresponds to $2\eta_1$ (from now on we will write η_j for $\eta(j)$). In the sequel, we will call Lévy-Loewner process a Loewner process driven by a function

$$\lambda(t) = e^{iL_t}$$

where L_t is a Lévy process with symbol η.

3.1 Expectation of $f_t(z)$ for Lévy-Loewner processes

The aim of this section is to give an explicit expression of the expectations of the coefficients a_n in the Lévy setting: as a corollary we will obtain expectations of the function f_t.

In order to simplify the computations we will write $b_n(t) = a_n(t)e^{-(n-1)t}$. If we identify the coefficients of the left-side of the Loewner equation to the right-side ones, one gets the recursion formula

$$\dot{b_n} = 2\sum_{k=1}^{n-1} kb_k e^{-(n-k)s} e^{-i(n-k)L_t}.$$

From this recursion formula one can extract the induction formula

$$\dot{b_n} = X_t \dot{b_{n-1}} + 2(n-1)b_{n-1}X_t,$$

where we have put

$$X_t = e^{-t-iL_t}.$$

To simplify further the computation let us use as the unknown $c_n = \dot{b_n}$ and put $\varphi_n(t) = Ec_n(t)$. The induction relation becomes

$$c_n = X_t c_{n-1} - 2(n-1)X_t \int_t^\infty c_{n-1}(s)ds. \tag{2}$$

Some experiments with the computation of the few first terms yields to the following induction formula for $\varphi_n(t)$:

$$\varphi_n(t) = e^{-(1+\eta_{n-1}-\eta_{n-2})t}\left(\varphi_{n-1}(t) - 2(n-1)\int_t^\infty \varphi_{n-1}(u)du\right).$$

One then easily find our way to the computation of $E(a_n) = -\int_0^\infty \varphi_{n-1}(u)du$:

$$E(a_n) = -2\frac{\prod_{j=3}^{n}(\eta_{j-2}-j)}{\prod_{j=1}^{n-1}(j+\eta_j)}.$$

In the case of SLE_κ this formula gives for the few first terms:

(I) $Ea_2 = -\dfrac{4}{2+\kappa}$,

(II) $Ea_3 = -\dfrac{\kappa-6}{(1+\kappa)(2+\kappa)}$,

(III) $Ea_4 = -\dfrac{4(\kappa-6)(\kappa-2)}{(6+9\kappa)(1+\kappa)(2+\kappa)}$.

In particular, we get for $\kappa = 6$,

$$Ef_t(z) = z - z^2/2 = \frac{1}{2}(1-(z-1)^2),$$

and, for $\kappa = 2$,

$$Ef_t(z) = z - z^2 + z^3/3 = \frac{1}{3}((z-1)^3+1).$$

More generally, $z \mapsto Ef_t(z)$ is polynomial for the sequence of values

$$\kappa = \frac{2n}{(n-2)^2},\ n \geq 3.$$

3.2 Computation of $E(|a_n|^2)$ for small n

Theorem 3.1. *For Lévy-Loewner processes we have*

$$E(|a_2|^2) = \Re\left(\frac{4}{1+\eta_1}\right).$$

Proof - We recall the expressions for a_2 and a_3:

$$\begin{aligned} a_2(t) &= -2e^t \int_t^\infty \overline{\lambda}(s)e^{-s}ds, \\ a_3(t) &= 4e^{2t}\left(\int_t^\infty \overline{\lambda}(s)e^{-s}ds\right)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds. \end{aligned}$$

We can thus write

$$|a_2|^2 = 8\int_0^\infty e^{-s+iL_s}\int_s^\infty e^{-s'-iL_{s'}}ds'ds = 8\int_0^\infty e^{-s}\int_s^\infty e^{-s'-i(L_{s'}-L_s)}ds'ds.$$

Using now the expression for the characteristic function of $L_s - L_{s'}$, we get

$$E(|a_2|^2) = 8\int_0^\infty e^{-s}\int_s^\infty e^{-s'-(s'-s)\eta_1}ds'ds,$$

and the result follows. □

We now pass to computations involving a_3. In order to avoid repetitions of computations needed for various consequences, we will compute

$$E(|a_3 - \mu a_2^2|^2)$$

where μ is a real constant. By the above computations,

$$a_3(t) = 4e^{2t}\left(\int_t^\infty \overline{\lambda}(s)e^{-s}ds\right)^2 - 2e^{2t}\int_t^\infty e^{-2s}\overline{\lambda}^2(s)ds.$$

We may then write

$$e^{-4t}|a_3 - \mu a_2^2|^2 = 16(1-\mu)^2 I_1 - 16(1-\mu)\Re I_2 + 4I_3,$$

where

$$\begin{aligned} I_1 &= \int_t^\infty\int_t^\infty\int_t^\infty\int_t^\infty e^{-(s_1+s_2+s_3+s_4)}\overline{\lambda}(s_1)\lambda(s_2)\overline{\lambda}(s_3)\lambda(s_4)ds_1ds_2ds_3ds_4, \\ I_2 &= \int_t^\infty\int_t^\infty\int_t^\infty e^{-(s_1+s_2+2s_3)}\overline{\lambda}(s_1)\overline{\lambda}(s_2)\lambda(s_3)^2ds_1ds_2ds_3, \\ I_3 &= \int_t^\infty\int_t^\infty e^{-2(s_1+s_2)}\overline{\lambda}(s_1)^2\lambda(s_2)^2ds_1ds_2. \end{aligned}$$

From now on we put $t = 0$ in the above formulas. The computation of I_3 follows the same lines as the one in the Theorem 3.1 and we find

$$E(I_3) = \Re\left(\frac{1}{2(2+\eta_2)}\right).$$

In order to compute $E(I_2)$ we have to use the strong Markov property. First, by symmetry, we may write

$$I_2 = 2\int_{s_1=0}^{\infty}\int_{s_2=s_1}^{\infty}\int_{s_3=0}^{\infty} e^{-(s_1+s_2+2s_3)}e^{i(L_{s_3}-L_{s_1})}e^{i(L_{s_3}-L_{s_2})}ds_1ds_2ds_3,$$

and we cut this integral as $I_2 = 2(I_{2,1} + I_{2,2} + I_{2,3})$ where in $I_{2,1}$ (resp. in $I_{2,2}, I_{2,3}$), s_3 ranges in $[0, s_1]$ (resp. in $[s_1, s_2], [s_2, \infty)$). For $I_{2,1}$ we write

$$e^{i(L_{s_3}-L_{s_1})}e^{i(L_{s_3}-L_{s_2})} = e^{-2i(L_{s_1}-L_{s_3})}e^{-i(L_{s_2}-L_{s_1})}$$

so that we can use Markov property and deduce that the expectation of this random variable is

$$e^{-\overline{\eta_1}(s_1-s_3)}e^{-\overline{\eta_1}(s_2-s_1)}.$$

From this the value of $E(I_{2,1})$ can be easily computed and we find

$$E(I_{2,1}) = \frac{1}{4(1+\overline{\eta_1})(2+\overline{\eta_2})}.$$

Similar considerations lead to

$$E(I_{2,2}) = \frac{1}{4(1+\overline{\eta_1})(3+\eta_1)},$$
$$E(I_{2,3}) = \frac{1}{4(2+\eta_2)(3+\eta_1)}.$$

Combining these computations we get

$$\Re(E(I_2)) = \Re\left(\frac{1}{2(1+\eta_1)(2+\eta_2)} + \frac{1}{2(1+\overline{\eta_1})(3+\eta_1)} + \frac{1}{2(2+\eta_2)(3+\eta_1)}\right).$$

The computation of I_1 follows the same lines. First, by symmetry,

$$I_1 = 4\int_0^{\infty}\int_{s_1}^{\infty}\int_0^{\infty}\int_{s_3}^{\infty} e^{-(s_1+s_2+s_3+s_4)}e^{i(L_{s_3}-L_{s_1})}e^{i(L_{s_4}-L_{s_2})}ds_1ds_2ds_3ds_4.$$

We then split this integral into a sum of six pieces according to:

(I) $s_3 < s_4 < s_1 < s_2$,

(II) $s_3 < s_1 < s_4 < s_2$,

(III) $s_3 < s_1 < s_2 < s_4$,

(IV) $s_1 < s_3 < s_4 < s_2$,

(V) $s_1 < s_3 < s_2 < s_4$,

(VI) $s_1 < s_2 < s_3 < s_4$.

Clearly (I) $= \overline{\text{(VI)}}$, (II) $= \overline{\text{(V)}}$ and (III) $= \overline{\text{(IV)}}$. Using the same arguments as in the previous computations, skipping the details, we get

$$\begin{aligned} E(\mathrm{I}) &= \frac{1}{4(1+\overline{\eta_1})(2+\overline{\eta_2})(3+\overline{\eta_1})}, \\ E(\mathrm{II}) &= \frac{1}{8(1+\overline{\eta_1})(3+\overline{\eta_1})}, \\ E(\mathrm{III}) &= \frac{1}{8(1+\eta_1)(3+\overline{\eta_1})}. \end{aligned}$$

Altogether we get

$$E(I_1) = \Re\left(\frac{2}{(1+\eta_1)(2+\eta_2)(3+\eta_1)} + \frac{1}{(1+\eta_1)(3+\eta_1)} + \frac{1}{(1+\overline{\eta_1})(3+\eta_1)}\right).$$

We may now state

Theorem 3.2. *If μ is a real coefficient then*

$$E(|a_3 - \mu a_2^2|^2) =$$
$$\Re\left(\frac{16(1-\mu)^2(4+\eta_2)}{(1+\eta_1)(2+\eta_2)(3+\eta_1)} - \frac{16(1-\mu)(2+\eta_1)}{(1+\eta_1)(2+\eta_2)(3+\eta_1)} + \frac{2}{2+\eta_2} + \frac{8(1-\mu)(1-2\mu)}{(\overline{\eta_1}+1)(\eta_1+3)}\right).$$

In the case η real and even, this becomes

$$E(|a_3 - \mu a_2^2|^2) = \Re\left(\frac{32(1-\mu)^2(3+\eta_2) - 8(1-\mu)(6+2\eta_1+\eta_2) + 2(1+\eta_1)(3+\eta_1)}{(1+\eta_1)(2+\eta_2)(3+\eta_1)}\right).$$

Finally, in the SLE case $\eta(\xi) = \frac{\kappa}{2}|\xi|^2$,

$$E(|a_3 - \mu a_2^2|^2) = \frac{(108 - 288\mu + 192\mu^2) + (88 - 208\mu + 128\mu^2)\kappa + \kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)}.$$

3.3 Some corollaries

The first corollary is the analogue of Loewner's estimate, i.e. the value obtained by taking $\mu = 0$.

Theorem 3.3. *For Lévy-Loewner processes with η real and even we have*

$$E(|a_3|^2) = \frac{1}{(1+\eta_1)(3+\eta_1)}\left[24 + 2\frac{(\eta_1-1)(\eta_1-3)}{2+\eta_2}\right].$$

In the case of SLE this reads

$$E(|a_3|^2) = \frac{108+88\kappa+\kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)}.$$

Notice the important role played by the cases $\eta_1 = 1, 3$, corresponding to $\kappa = 2, 6$; in these cases the result does not depend on η_2, and is, respectively, 3 and 1.

The second corollary shows that there is no Fekete-Szego counter-example in the SLE family. We start with $f \in S$ being f_0 for a SLE_κ process. We associate to it the odd function h as above, that is $h(z) = z\sqrt{\frac{f(z)}{z}} = z + b_3 z^3 + b_5 z^5 + \dots$ while $f(z) = z + a_2 z^2 + a_3 z^3 + \dots$, an easy computation gives

$$b_5 = \frac{1}{2}(a_3 - \frac{1}{4}a_2^2).$$

We thus put $\mu = \frac{1}{4}$ in the above theorem and get

$$E(|b_5|^2) = \Re\left(\frac{18+9\eta_2-4\eta_1+2\eta_1^2}{4(1+\eta_1)(2+\eta_2)(3+\eta_1)} + \frac{3}{4}\frac{1}{(1+\overline{\eta_1})(3+\eta_1)}\right).$$

In the case of η real and even, we have

$$E(|b_5|^2) = \Re\left(\frac{6+3\eta_2-\eta_1+\eta_1^2/2}{(1+\eta_1)(3+\eta_1)(2+\eta_2)}\right).$$

Finally, in the $SLE-$case,

$$E(|b_5|^2) = \frac{12+44\kappa+\kappa^2}{(1+\kappa)(2+\kappa)(6+\kappa)},$$

a value which is always less than or equal to 1 (the equality holds for $\kappa = 0$).

The last corollary concerns the schwarzian derivative, whose definition is

$$S_f(z) = \frac{f'''(0)}{f'(0)} - \frac{3}{2}\left(\frac{f''(0)}{f'(0)}\right)^2.$$

We easily obtain $S_f(0) = 6(a_3 - a_2)^2$, thus corresponding to $\mu = 1$. The result is

$$E(|S_f(0)|^2) = \frac{12}{2+\eta_2},$$

and in the SLE-case,

$$E(|S_f(0)|^2) = \frac{36}{1+\kappa}.$$

A few comments about these results:
- It is striking that $E(|a_2|^2) = E(|a_3|^2) = 1$ for $\kappa = 6$. We will come back to this result in the next section where we perform some computer experiments;
- For all values of κ we have $E(|b_5|^2) \leq 1$: there is no Fekete-Szego counterexample in the SLE-family. Using Schoenberg property of the Lévy symbol η it can also be seen that ther is no Fekete-Szego counterexample at the expectation level for general Lévy-Loewner processes for η real and even. Does it remain for higher order terms or for higher moments? This is not clear since the formulas are complicated and that it is not clear if the values of the expectations are decreasing as a function of κ;
- It is known that $|S_f(0)|^2 \leq 6$ whenever f is injective. Conversely, if

$$(1-|z|^2)|S_f(z)|^2 \leq 2$$

then f is injective; in our case the value of 2 is reached for $\kappa \geq 8$. What is the interpretation of this fact?

3.4 Computer experiments

As we may see, these computations are quite involved, and it is clear that they will become exponentially more complicated. Moreover, its seems impossible to find a closed formula for all the terms. The rest of this section is devoted to the description of an algorithm that we have implemented on MATLAB to compute $E(|a_n|^2)$. This algorithm is divided into two parts: the first encodes the computation of a_n, while the second uses it to compute $E(|a_n|^2)$. Because SLE and α-stable processes are Lévy processes with Lévy symbol η real and even, we restrict to this case.

For the encoding of a_n we use (2), this allows us to write the $a_n's$ as linear combinations of integrals of the form

$$\int_t^\infty e^{-i\alpha_1 L_{s_1} - \beta_1 s_1} \int_{s_1}^\infty e^{-i\alpha_1 L_{s_2} - \beta_2 s_2} \ldots \int_{s_{k-1}}^\infty e^{-i\alpha_k L_{s_k} - \beta_k s_k} ds_1 \ldots ds_k$$

which will be encoded as

$$(\alpha_1, \beta_1) \ldots (\alpha_k, \beta_k) \quad (1 \leq k \leq n).$$

These integrals can be explicitely computed by using as above the strong Markov property and the value of the characteristic function of normal laws and the result is:

$$(\alpha_1,\beta_1)\dots(\alpha_k,\beta_k)=\prod_{j=0}^{k-1}\left[\beta_k+\beta_{k-1}+\dots+\beta_{k-j}+\eta(\alpha_k+\alpha_{k-1}+\dots+\alpha_{k-j})\right]^{-1}.$$

When we next compute $|a_n|^2$ we need to compute products of such integrals with complex conjugate of others, that we symbolically denote by

$$[(\alpha_1,\beta_1)\dots(\alpha_k,\beta_k);(-\alpha_1',\beta_1')\dots(-\alpha_l',\beta_l')]\quad(1\le k,l\le n).$$

Such a product may be written as a sum of $\binom{k+l}{k}$ integrals with $k+l$ variables: the k first and the l last are ordered and the number of integrals corresponds to the number of ways of shuffling k cards in the left hand with l cards in the right hand.

This sum is enormous and, in order to accurately compute it, we write it as a sum of integrals of the form (2) starting by (α_1,β_1) with those starting by $(-\alpha_1',\beta_1')$, thus reducing the work to a computation at lower order. Using dynamic programing we can perform computations at order $n\le 20$ on a usual computer. Here are the results for a_3, a_4 and a_5 in the case of Lévy processes:

$$E(|a_3|^2)=\frac{3!2^2}{(\eta_1+1)(\eta_1+3)}+\frac{2(\eta_1-1)(\eta_1-3)}{(\eta_1+1)(\eta_1+3)(\eta_2+2)};$$

$$\begin{aligned}E\left(|a_4|^2\right)=&\frac{4!2^3}{(\eta_1+1)(\eta_1+3)(\eta_1+5)}+\frac{4(\eta_1-1)(\eta_1-3)\eta_2(\eta_2-4)(\eta_1+3)}{3(\eta_1+1)(\eta_1+3)(\eta_1+5)(\eta_2+2)(\eta_2+4)(\eta_3+3)}\\&+\frac{32(\eta_1-1)(\eta_1-3)}{(\eta_1+1)(\eta_1+3)(\eta_1+5)(\eta_2+2)(\eta_2+4)};\end{aligned}$$

$$\begin{aligned}E\left(|a_5|^2\right)=&\frac{5!2^4}{(\eta_1+1)(\eta_1+3)(\eta_1+5)(\eta_1+7)}\\&+\frac{4(\eta_1-1)(\eta_1-3)\eta_2(\eta_2-4)(\eta_1+3)(\eta_3+1)(\eta_3-5)(\eta_1+3)(\eta_1+5)(\eta_2+4)}{3(\eta_1+1)(\eta_1+3)(\eta_1+5)(\eta_1+7)(\eta_2+2)(\eta_2+4)(\eta_2+6)(\eta_3+3)(\eta_3+5)(\eta_4+4)}\\&+\frac{Q(\eta_1-1)(\eta_1-3)}{(\eta_1+1)(\eta_1+3)(\eta_1+5)(\eta_1+7)(\eta_2+2)(\eta_2+4)(\eta_2+6)(\eta_3+3)(\eta_3+5)}.\end{aligned}$$

where

$$\begin{aligned}Q=&\frac{4}{3}(24\eta_1^2\eta_2^2+9\eta_1^2\eta_2\eta_3^2+72\eta_1^2\eta_2\eta_3+39\eta_1^2\eta_2+36\eta_1^2\eta_3^2+288\eta_1^2\eta_3+520\eta_1^2+19\eta_1\eta_2^3\eta_3\\&+77\eta_1\eta_2^3+56\eta_1\eta_2^2\eta_3+472\eta_1\eta_2^2-36\eta_1\eta_2\eta_3^2-816\eta_1\eta_2\eta_3-3660\eta_1\eta_2-144\eta_1\eta_3^2\\&-1152\eta_1\eta_3-2160\eta_1+75\eta_2^3\eta_3+285\eta_2^3+348\eta_2^2\eta_3^2+2952\eta_2^2\eta_3+6420\eta_2^2+3507\eta_2\eta_3^2\\&+26184\eta_2\eta_3+43245\eta_2+8460\eta_3^2+67680\eta_3+126900).\end{aligned}$$

We will end this section with the results for a_4, a_5, a_6, a_7 and a_8 in the SLE-case:

$$E\left(|a_4|^2\right) = \frac{8}{9}\frac{\kappa^5 + 104\kappa^4 + 4576\kappa^3 + 18288\kappa^2 + 22896\kappa + 8640}{(\kappa+10)(3\kappa+2)(\kappa+6)(\kappa+1)(\kappa+2)^2}\,;$$

$$\begin{aligned}E\left(|a_5|^2\right) = &(27\kappa^8 + 3242\kappa^7 + 194336\kappa^6 + 6142312\kappa^5 + 42644896\kappa^4\\ &+ 119492832\kappa^3 + 153156096\kappa^2 + 87882624\kappa + 18144000)\\ &/[36(\kappa+14)(3\kappa+2)(\kappa+10)(2\kappa+1)(\kappa+6)(\kappa+3)(\kappa+1)(\kappa+2)^2]\,;\end{aligned}$$

$$\begin{aligned}E\left(|a_6|^2\right) = &\frac{2}{225}(216\kappa^{10} + 29563\kappa^9 + 2062556\kappa^8 + 90749820\kappa^7 + 2277912280\kappa^6\\ &+ 16419864848\kappa^5 + 50825787744\kappa^4 + 76716664128\kappa^3\\ &+ 58263304320\kappa^2 + 21233664000\kappa + 2939328000)\\ &/[(\kappa+18)(3\kappa+2)(\kappa+14)(2\kappa+1)(\kappa+10)(\kappa+6)(5\kappa+2)\\ &(\kappa+3)(\kappa+1)(\kappa+2)^2]\,;\end{aligned}$$

$$\begin{aligned}E\left(|a_7|^2\right) = &\frac{1}{8100}(27000\kappa^{15} + 4479353\kappa^{14} + 373838334\kappa^{13} + 20594712527\kappa^{12}\\ &+ 787796136854\kappa^{11} + 19121503739240\kappa^{10} + 221861771218136\kappa^9\\ &+ 1386550697705712\kappa^8 + 5130607642056896\kappa^7 + 11854768997862912\kappa^6\\ &+ 17547915006086400\kappa^5 + 16725481436226816\kappa^4 + 10110569026936320\kappa^3\\ &+ 3711483045734400\kappa^2 + 749049576192000\kappa + 63371911680000)\\ &/[(\kappa+22)(3\kappa+1)(5\kappa+2)(\kappa+18)(2\kappa+1)(\kappa+14)(3\kappa+2)\\ &(\kappa+10)(\kappa+6)(\kappa+5)(\kappa+3)(\kappa+1)^2(\kappa+2)^3]\,;\end{aligned}$$

$$\begin{aligned}E\left(|a_8|^2\right) = &\frac{2}{99225}(729000\kappa^{18} + 143757261\kappa^{17} + 14031668642\kappa^{16} + 906444920407\kappa^{15}\\ &+ 42715714646750\kappa^{14} + 1476227672190480\kappa^{13} + 34674813906653712\kappa^{12}\\ &+ 471116720002819536\kappa^{11} + 3802657434377773600\kappa^{10}\\ &+ 19218418658636100992\kappa^9 + 63191729416067875840\kappa^8\\ &+ 138392538501661946112\kappa^7 + 204258207932541043200\kappa^6\\ &+ 203508494170475323392\kappa^5 + 135640094878259859456\kappa^4\\ &+ 59063686024095313920\kappa^3 + 16005106174366310400\kappa^2\\ &+ 2435069931098112000\kappa + 158176291553280000)\\ &/[(7\kappa+2)(5\kappa+2)(\kappa+26)(3\kappa+1)(\kappa+22)(2\kappa+1)(\kappa+18)(\kappa+14)\\ &(3\kappa+2)(\kappa+10)(\kappa+5)(\kappa+3)(\kappa+6)^2(\kappa+1)^2(\kappa+2)^3]\,.\end{aligned}$$

3.5 Interpretation of the results

The fact that $E(|a_n|^2) = 1$ for $\kappa = 6$ is proven for $n = \overline{2,8}$. We of course predict that this is true for all values of n but we do not know how to prove it yet. If we admit this prediction we would get the following corollary, using Plancherel theorem:

Corollary 3.1. *For a Lévy process with $\eta_1 = 0, 1, 3$ ($\kappa = 0, 2, 6$) respectively,*

$$E\left(\frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta\right) = \frac{1+7r^2+r^4-r^6}{(1-r^2)^5}, = \frac{1+2r^2-r^4}{(1-r^2)^4}, = \frac{1+r^2}{(1-r^2)^3}$$

respectively.

We can rephrase this corollary in terms of integral means spectrum.

Definition 3.1. *The integral means spectrum of the conformal mapping f is the function defined on $\mathbb{R}$ by*

$$\beta(p) = \overline{\lim}_{r\to 1} \frac{\log(\int_{\partial\mathbb{D}} |f'(rz)|^p |dz|)}{\log(\frac{1}{1-r})}.$$

This spectrum is related to the other multifractal spectra.

The preceeding results show that in expectation and in a very strong sense,

$$\beta(2) = 5, 4, 3$$

if $\eta_1 = 0, 1, 3$ ($\kappa = 0, 2, 6$) respectively. Another interesting random variable is the area of the image of the disk, i.e.

$$\int\int_{\mathbb{D}} |f'(z)|^2 dxdy = \pi \sum_1^\infty n|a_n|^2.$$

Assuming the validity of the above prediction the expectation of this quantity is infinite for $\kappa \leq 6$. This would mean that, even if after $\kappa = 4$ the SLE trace is no longer a simple curve, this curve does not turn around 0 at least for $\kappa \leq 6$. Numerical experiments are hard to detect if the series converge for $\kappa = 6$ but the first partial computations seem to indicate that it is indeed the case.

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